Math 60 10.3 Solving Equations that are Quadratic in Form - 1st

Objectives

1) Solve eqns having $(\text{stuff})^2$ and (stuff) using u-substitution $u = \text{stuff}$, $u^2 = (\text{stuff})^2$

Examples:
- $\text{stuff} = x^2$ $\quad (\text{stuff})^2 = (x^2)^2 = x^4$
- $\text{stuff} = \sqrt{x}$ $\quad (\text{stuff})^2 = (\sqrt{x})^2 = x$
- $\text{stuff} = x^{-1}$ $\quad (\text{stuff})^2 = (x^{-1})^2 = x^{-2}$
- $\text{stuff} = (z^2+3)$ $\quad (\text{stuff})^2 = (z^2+3)^2$
- $\text{stuff} = x^{1/3}$ $\quad (\text{stuff})^2 = (x^{1/3})^2 = x^{2/3}$
- $\text{stuff} = x^{1/4}$ $\quad (\text{stuff})^2 = (x^{1/4})^2 = x^{1/2}$

The equations in this section are <u>not</u> true quadratic equations $ax^2+bx+c=0$.

<u>But</u>: We can make a temporary substitution, replacing

$$\left.\begin{aligned} (\text{stuff}) &= u \\ \text{and} \quad (\text{stuff})^2 &= u^2 \end{aligned}\right\} \quad \text{in } a(\text{stuff})^2 + b(\text{stuff}) + c = 0$$

to get a quadratic $au^2 + bu + c = 0$.

An equation $a(\text{stuff})^2 + b(\text{stuff}) + c = 0$ is said to be <u>quadratic in form</u>, because its form or structure is essentially quadratic.

In $ax^2+bx+c=0$, the a, b and c are numbers. Similarly, the a, b and c in $a(\text{stuff})^2 + b(\text{stuff}) + c = 0$ are also numbers -- the coefficients.

<u>Basic steps in the process:</u>

1) Recognize quadratic form and identify $u = \text{stuff}$, $u^2 = (\text{stuff})^2$
2) Substitute u for (stuff) and write quadratic $au^2+bu+c=0$.
3) Solve the quadratic $au^2+bu+c=0$ using - square root property - factoring or - CTS or QF.
   to get $u=\#$, $u=\#$
4) For each $u=\#$, replace u by (stuff). $(\text{stuff}) = \#$
5) Solve $(\text{stuff}) = \#$ by isolating the original variables.

Math 60 10.3 − 1st

① Solve $q^4 + 15 = 8q^2$

To recognize quadratic form, set $= 0$ in standard form.

$q^4 - 8q^2 + 15 = 0.$

u is often (though not always) the variable part of the middle term.

$\underline{\underline{u = q^2}}$

check by squaring both sides $u^2 = (q^2)^2 = q^4$

should get variable part of 1st term ☺

Rewrite by substituting $u = q^2$ (stuff $= q^2$ in this case)

$u^2 - 8u + 15 = 0$

Solve quadratic.

$(u-5)(u-3) = 0$

factoring: 15, −5, −3, −8

$u = 5 \qquad u = 3$

Replace u by stuff, u replaced by q^2: $\underline{\underline{u = q^2}}$

$q^2 = 5 \qquad q^2 = 3$

Solve for original variables:

$\boxed{q = \pm\sqrt{5} \qquad q = \pm\sqrt{3}}$

Math 60 10.3 – 1st

(2) Solve $x - 3\sqrt{x} - 4 = 0$

<u>option 1</u>: Substitute $u = \sqrt{x}$

<u>option 2</u>: isolate $\sqrt{\ }$ and square both sides

<u>option 1</u>: $\underline{\underline{u = \sqrt{x}}}$ $\quad u^2 = (\sqrt{x})^2 = x$

$u^2 - 3u - 4 = 0$

$(u-4)(u+1) = 0$

$u = 4 \qquad u = -1$

$\sqrt{x} = 4 \qquad \sqrt{x} = -1$

$(\sqrt{x})^2 = (4)^2 \qquad (\sqrt{x})^2 = (-1)^2$

$x = 16 \qquad x = 1$

were you wide awake when we wrote $\sqrt{x} = -1$?

check for extraneous

$x = 16$: $\quad x - 3\sqrt{x} - 4 \overset{?}{=} 0$

$16 - 3\sqrt{16} - 4 = 0$

$16 - 3(4) - 4 = 0$

$16 - 12 - 4 = 0$ ✓

$X = +1 \quad +1 - 3\sqrt{+1} - 4 = 0$

$+1 - 3 - 4 \neq 0$ no.

$\boxed{x = 16}$

<u>option 2</u>: $x - 3\sqrt{x} - 4 = 0$

$x - 4 =$

$(x-4)^2 = (3\sqrt{x})^2$

$x^2 - 8x + 16 = 9x$

$x^2 - 17x + 16 = 0$

$(x-16)(x-1) = 0$

$\boxed{x = 16}$ $\cancel{x = 1}$

check for extraneous, as above

Math 60 10.3 – 1st

yes (3) Solve $3a^{-2} - 10a^{-1} - 8 = 0$

option 1: use quadratic form by substituting

$u = a^{-1}$

$u^2 = (a^{-1})^2 = a^{-2}$

option 2: Write negative exp as positive exp in denominators, then multiply by LCD (c7 method).

option 1: $3a^{-2} - 10a^{-1} - 8 = 0$

$u = a^{-1}$ substitute $\underline{u = a^{-1}}$

$u^2 = (a^{-1})^2 = a^{-2}$

$3u^2 - 10u - 8 = 0$

$3(-8)$
-24
$12 \quad -2$
10

$D = b^2 - 4ac$
$= (-10)^2 - 4(3)(-8)$
$= 100 + 96$
$= 196$
$\sqrt{196} = 14$ so it does factor!

$3u^2 + 12u - 2u - 8 = 0$

$3u(u+4) - 2(u+4) = 0$

$(u+4)(3u-2) = 0$

$u + 4 = 0 \qquad 3u - 2 = 0$

$u = -4 \qquad u = \frac{2}{3}$

substitute back

$a^{-1} = -4 \qquad a^{-1} = \frac{2}{3}$

Hint: short cut is to take reciprocals of both sides.

$\frac{1}{a} = \frac{-4}{1} \qquad \frac{1}{a} = \frac{2}{3}$

cross-multiply

notice $a \neq 0$, so extraneous is possible, though unlikely

$1 = -4a \qquad 3 = 2a$

$\boxed{-\frac{1}{4} = a} \qquad \boxed{\frac{3}{2} = a}$

Math 60 10.3 - 1st

option 2: $3a^{-2} - 10a^{-1} - 8 = 0$

$$\frac{3}{a^2} - \frac{10}{a} - 8 = 0 \qquad LCD = a^2$$

$$\frac{3}{a^2} \cdot \frac{a^2}{1} - \frac{10}{a} \cdot \frac{a^2}{1} - \frac{8}{1} \cdot \frac{a^2}{1} = 0 \cdot \frac{a^2}{1}$$

mult all terms both Sides by LCD,

$$3 - 10a - 8a^2 = 0$$

$$0 = 8a^2 + 10a - 3$$

$8(-3)$

-24

$12 \quad -2$

10

$$\underbrace{8a^2 - 2a}\ \underbrace{+12a - 3} = 0$$

$$2a(4a-1) + 3(4a-1) = 0$$

$$(4a-1)(2a+3) = 0$$

$$\boxed{a = \frac{1}{4}} \qquad \boxed{a = -\frac{3}{2}}$$

$D = b^2 - 4ac$

$= 10^2 - 4(8)(-3)$

$= 100 + 96$

$= 196$

$\sqrt{196} = 14$, it factors

Math 60 10.3 – 1st

4 Solve $(z^2+3)^2 - 2(z^2+3) - 8 = 0$

Solve $\underbrace{(z^2+3)^2}_{(\text{stuff})^2} - 2\underbrace{(z^2+3)}_{(\text{stuff})} - 8 = 0$

substitute $\underline{u = z^2+3}$

$u^2 = (z^2+3)^2$

$u^2 - 2u - 8 = 0$

$(u-4)(u+2) = 0$ factor

$u = 4 \qquad u = -2$ solve

$\downarrow \qquad \downarrow$

$z^2+3 = 4 \qquad z^2+3 = -2$ replace u by z^2+3

$z^2 = 1 \qquad z^2 = -5$

$z = \pm\sqrt{1} \qquad z = \pm\sqrt{-5}$

$\boxed{z = \pm 1} \qquad \boxed{z = \pm\sqrt{5}\,i}$

Yes, you could solve #4 by FOIL, dist, combine, but the algebra is long and frought with opportunities to screw up. So please don't.

Math 60 10.3 – 1st

5 Solve $\left(\frac{1}{x+2}\right)^2 + \frac{6}{x+2} = 7$

Option 1:

$$\left(\frac{1}{x+2}\right)^2 + 6\cdot\left(\frac{1}{x+2}\right) - 7 = 0$$

Identify stuff and (stuff)^2. Set = 0.

Let $u = \frac{1}{x+2}$

$u^2 = \left(\frac{1}{x+2}\right)^2$

Note: $x \neq -2$ because it would cause $\div 0$

$u^2 + 6u - 7 = 0$ factor

$(u+7)(u-1) = 0$ solve

$u = -7 \qquad u = 1$ replace u by $\frac{1}{x+2}$

$\frac{1}{x+2} = -7 \qquad \frac{1}{x+2} = 1$ mult by LCD / cross-multiply

$1 = -7(x+2) \qquad 1 = 1(x+2)$

$1 = -7x - 14 \qquad 1 = x+2$

$15 = -7x \qquad \boxed{-1 = x}$

$\boxed{-\frac{15}{7} = x}$

Neither answer is $x = -2$, so no extraneous answers.

Option 2:

Yes, you could multiply the equation by $(x+2)^2 = \text{LCD}$

$$\left(\frac{1}{x+2}\right)^2 + \frac{6}{x+2} - 7 = 0$$

$$\frac{1}{\cancel{(x+2)^2}}\cdot\frac{\cancel{(x+2)^2}}{1} + \frac{6}{\cancel{(x+2)}}\cdot(x+2)^{\cancel{2}} - 7\cdot(x+2)^2 = 0\cdot(x+2)^2$$

$$1 + 6(x+2) - 7(x+2)^2 = 0$$

Then use $u = x+2$ and $u = (x+2)^2$.

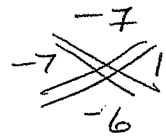
Math 60 10.3 – 1st

$$1 + 6(x+2) - 7(x+2)^2 = 0$$

rearrange to standard form

$$0 = 7(x+2)^2 - 6(x+2) - 1$$

$$0 = 7u^2 - 6u - 1$$



$$0 = 7u^2 - 7u + u - 1$$

$$0 = 7u(u-1) + 1(u-1)$$

$$0 = (u-1)(7u+1)$$

$$u = 1 \qquad u = -\frac{1}{7}$$

Replace u by $x+2$:

$$x+2 = 1 \qquad x+2 = -\frac{1}{7}$$

$$\boxed{x = -1} \qquad x = -\frac{1}{7} - 2$$

$$x = -\frac{1}{7} - \frac{14}{7}$$

$$\boxed{x = \frac{-15}{7}}$$

Math 60 10.3 – 1st

6 Solve $a^{2/3} + 3a^{1/3} - 28 = 0$

Notice $u = a^{1/3}$

$u^2 = (a^{1/3})^2 = a^{2/3}$ multiply exponents

Substitute: $u^2 + 3u - 28 = 0$

$(u+7)(u-4) = 0$ factor

$u = -7 \quad u = 4$ solve

$a^{1/3} = -7 \quad a^{1/3} = 4$ replace u by $a^{1/3}$

$(a^{1/3})^3 = (-7)^3 \quad (a^{1/3})^3 = 4^3$ cube both sides

$\boxed{a = -343} \quad \boxed{a = 64}$

CAUTION Do not try to cube both sides of this equation -- you must use ( ) on entire side and FOIL, which creates disastrous results.

Math 60 10.3 - 2nd

DO NOT DO THIS!!

$$\cancel{a^{2/3} + 3a^{1/3} - 28 = 0}$$

$$\cancel{\sqrt[3]{a^2} + 3\sqrt[3]{a} - 28 = 0}$$

Isolate $\sqrt[3]{a}$

$$\cancel{3\sqrt[3]{a} = 28 - \sqrt[3]{a^2}}$$

$$\cancel{(3\sqrt[3]{a})^3 = (28 - \sqrt[3]{a^2})^3}$$

cube both sides
* MUST use ( ).

$$\cancel{27a = \underbrace{\underbrace{(28 - \sqrt[3]{a^2})(28 - \sqrt[3]{a^2})}_{\text{FOIL this}}(28 - \sqrt[3]{a^2})}_{\text{then multiply again.}}}$$

OK, not too bad

uh-oh. It gets worse and worse and trust me......

IT NEVER GETS BETTER. ☹ ☹ ☹.

CAUTION

Absolutely do not cube each term separately:

Ex: $2 + 3 = 5$ true statement

$(2+3)^3 = 5^3$ true statement

Ex. $2^3 + 3^3 \neq 5^3$

$8 + 27 \neq 125$ false statement.

So: $\sqrt[3]{a^2} + 3\sqrt[3]{a} - 28 = 0$

DOES NOT MEAN

$$\cancel{(\sqrt[3]{a^2})^3 + (3\sqrt[3]{a})^3 - (28)^3 = 0}$$

Moral: For $2/3$ and $1/3$ powers, you have No choice. You must use u-substitution.

Math 60 10.3 Extra examples

① Solve $x^4+x^2-12=0$.

Hint: Double-underline so we can find it when we need it later

substitute $\underline{\underline{x^2=u}}$

this means $x^4=(x^2)^2=u^2$

$u^2+u-12=0$ ⟸ THIS IS A TRUE QUADRATIC.

$(u+4)(u-3)=0$ Let's solve by factoring

$u=-4 \quad u=3$

But these answers are values of u, and we want x

Recall that $u=x^2$, so we substitute this back.

$u=-4 \quad\quad u=3$

$\downarrow \quad\quad \downarrow$

$x^2=-4 \quad\quad x^2=3$

Solve again,

$x=\pm\sqrt{-4}$

$\boxed{x=\pm 2i}$

$\boxed{x=\pm\sqrt{3}}$

Note: Since the question is degree $\underline{\underline{4}}$, it makes sense that we have 4 solutions.

MAIN THEME OF 10.3:

substitute $u=$ stuff

and $u^2=(\text{stuff})^2$

M60 10.3 - 1st Extra Examples

(8) Solve $2x - 5\sqrt{x} + 2 = 0$.

Option 1: Use quadratic form by substituting

$$u = \sqrt{x}$$

$$x = (\sqrt{x})^2 = u^2$$

Option 2: Isolate $\sqrt{x}$ and square both sides (c9 method)

Option 1: $2u^2 - 5u + 2 = 0$ $\qquad \underline{\underline{u = \sqrt{x}}}$

$2(2) = 4$; -4, -1; -5

$2u^2 - 4u - u + 2 = 0$ $\qquad$ factor

$2u(u-2) - 1(u-2) = 0$

$(u-2)(2u-1) = 0$ $\qquad$ set factors $= 0$

$u - 2 = 0$ $\qquad$ $2u - 1 = 0$

$u = 2$ $\qquad$ $2u = 1$ $\qquad$ solve for u

$\downarrow$ $\qquad$ $u = \frac{1}{2}$

$\qquad\qquad$ $\downarrow$

$\sqrt{x} = 2$ $\qquad$ $\sqrt{x} = \frac{1}{2}$ $\qquad$ replace u by $\sqrt{x}$

$(\sqrt{x})^2 = 2^2$ $\qquad$ $(\sqrt{x})^2 = \left(\frac{1}{2}\right)^2$ $\qquad$ square both sides

$x = 4$ $\qquad$ $x = \frac{1}{4}$

check for extraneous: (since we squared both sides)

$2(4) - 5\sqrt{4} + 2 \stackrel{?}{=} 0$

$8 - 5(2) + 2 = 0$ ✓

$2\left(\frac{1}{4}\right) - 5\sqrt{\frac{1}{4}} + 2 \stackrel{?}{=} 0$

$\frac{1}{2} - 5 \cdot \frac{1}{2} + 2 = 0$ $\qquad$ $\boxed{x = 4, \frac{1}{4}}$

$\frac{1}{2} - \frac{5}{2} + \frac{4}{2} = 0$ ✓

M60 10.3 – 1st Extra Examples

Option 2: $2x - 5\sqrt{x} + 2 = 0$

$2x + 2 = 5\sqrt{x}$ isolate $\sqrt{x}$ and its coefficient

$(2x+2)^2 = (5\sqrt{x})^2$ Square both sides

$(2x+2)(2x+2) = 25 \cdot x$ FOIL LHS

$4x^2 + 8x + 4 = 25x$ Set $= 0$

$4x^2 - 17x + 4 = 0$

4(4)
16
−16 −1
−17

$D = b^2 - 4ac$
$= (-17)^2 - 4(4)(4)$
$= 225$

$\sqrt{225} = 15$ Perf sq, so it does factor!

$4x^2 - 16x - x + 4 = 0$

$4x(x-4) - 1(x-4) = 0$

$(x-4)(4x-1) = 0$

$\boxed{x = 4 \qquad x = \frac{1}{4}}$

check for extraneous (see work at end of option 1).

Math 60 10.3-1st Extra Examples

(9) Solve $\frac{1}{x^2} - \frac{5}{x} + 6 = 0.$

option 1: Substitute $u = \frac{1}{x} = x^{-1}$

$u^2 = \frac{1}{x^2} = x^{-2}$

option 1a: use positive exp in denominators

option 1b: use neg exponents in numerator

option 2: mult all by $LCD = x^2$ to clear fractions. (c7)

option 1a: $\frac{1}{x^2} - \frac{5}{x} + 6 = 0$

$u = \frac{1}{x}$

$\frac{1}{x^2} - 5 \cdot \frac{1}{x} + 6 = 0$

$u^2 - 5u + 6 = 0$ subst

$(u-3)(u-2) = 0$ factor

$u = 3 \quad u = 2$ solve

$\frac{1}{x} = 3 \quad \frac{1}{x} = 2$ subst $\frac{1}{x}$ for u

Math 60 10.3-1st

Solve $\frac{1}{x} = \frac{3}{1}$ $\qquad$ $\frac{1}{x} = \frac{2}{1}$ $\qquad$ cross-multiply

$1 = 3x$ $\qquad$ $1 = 2x$

$$\boxed{\frac{1}{3} = x \qquad \frac{1}{2} = x}$$

$x \neq 0$ so
not extraneous

<u>Option 1b:</u> $\frac{1}{x^2} - \frac{5}{x} + 6 = 0$

$x^{-2} - 5x^{-1} + 6 = 0$ $\qquad$ $u = x^{-1}$

$u^2 - 5u + 6 = 0$

then same as 1a to the end.

<u>option 2:</u> $\frac{1}{x^2} - \frac{5}{x} + 6 = 0$ $\qquad$ LCD $= x^2$

$\frac{1}{x^2} \cdot \frac{x^2}{1} - \frac{5}{x} \cdot \frac{x^2}{1} + \frac{6}{1} \cdot \frac{x^2}{1} = \frac{0}{1} \cdot \frac{x^2}{1}$ $\qquad$ mult all by x^2

$1 - 5x + 6x^2 = 0$

$6x^2 - 5x + 1 = 0$ $\qquad$ rearrange to standard form

$6 \cdot 1$
6
2 3
5

$\underbrace{6x^2 - 3x}\ \underbrace{-2x + 1} = 0$ $\qquad$ solve by factoring

$3x(2x-1) - 1(2x-1) = 0$

$(2x-1)(3x-1) = 0$

$$\boxed{x = \frac{1}{2} \qquad x = \frac{1}{3}}$$